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General Steps in the Revolution of the Calculus from the Time of the Ancients to the Present

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in the

Evolution of the Calculus

from the

Time of the Ancients

to the

Present

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General Steps in the Evolution of the Calculus
from the Time of the Ancients to the Present

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General Steps in the Evolution of the Calculus

the Time of the Ancients to the Present

Mathematics is the most ancient of the sciences, yet it is not surpassed by any in modernity , bu is flourishing to-day at a rate unsurpassed and unparalleled by means of the Calculus . Mathematics is like to a wheel , which has influenced mechanism .... Who invented this wheel , is not known but its influence is unconsciously felt by you and me, and the whole world about us in a greater or less degree .

How many of us do you think have stopped to consider the wonderful power of mathematics ; its usefulness in matters of everyday life ; its influence in matters of thought its necessity in the simplest as well as the most complex calculation ; its dominion over all science and art ?

Pure science , virtually without exception , depends upon mathematics . We cannot arrive at exact scientific results without accurate measurements . Things must be weighed, counted , scaled , measured , balanced against other combinations and aggregations which themselves must be measured. There was a time in the history of science when every thing was measured roughly . Physics , astronomy , chemistry , the calendar ,......everything was measured with a certain degree of approximation . Scientists were content with these reasonably accurate results , they scarcely hoped for anythin better. Their instruments were crude , their methods of measurement were imperfect ; they got what they could in the way of results and went their way rejoicing , mobe or less. To-day all this is changed ; accuracy is the indispensable criterion for all scientific work. Instruments must be accurate to an almost inconceivable degree of refinement . Perhaps thw choice between two scientific theories may depend upon a measurement involving a millionth of an inch .

The knowledge of mathematical principles is necessary to good seamanship, but perhaps in no art is the practical and actual handling of apparatus more useful than in that of the mariner . Theory can but lead the learner to the edge of the subject, science and practice must go hand in hand before any substantial acquirement can be gained . Numbers form the connecting link between theory and the application of theory to practical arts . In every mathematical principle ... mathematical formulae are implied , though they be extremely simple . It is for the mathematician to find out how far experimental confirmation of a theory can be pushed and where an hypothesis is necessary . Facts apparently unconnected are found to have their origin in a common source and often only the mathematician can trace their connection . More than this, the mathematician is able to draw corollaries and secondary truths from a given principle which the experimentalist alone does not discover. The significance of science, and the wonderful results it achieves with apparently simple means, are among the marvels of the modern age.

Modern mathematics begins with the invention of Analytic Geometry and the Calculus. The Calculus is one of the most powerful instruments of human thought. Beyond doubt the calculus is the greatest contribution ever made to the science of mathematics. It is the first and only mathematical subject defined by the article " The ", The Calculus. Before further discussion and exploration into the history of its development, let us take a look at the etymology and formal meaning of the term. Calculus... from the Latin " Calculatus" the past participle of calculare, i.e. to ascertain, to calculate or to determine by mathematical processes, usually by the ordinary rules of arithmetic. Hence in general it is a term applied in mathematics to any method of treating problems by means of a system of algebraic notation, a method of scientific computation which is applied in many regions of investigation; but is commonly taught as the highest branch of pure mathematics. As commonly used, the term refers particularly to infinitesimal, or differential and integral calculus, and deals with the properties of variable quantities. The rate of change of a variable is known as the differential, and the process of finding this differential when the variable is known is called integral calculus. Application of the principles and formulas so developed are made in many mechanical problems .

The Infinitesimal , or Differential and Integral Calculus is not so much a branch of mathematics as a method or instrument of mathematical investigation , of indefinite applicability . We might define calculus as the Theory and Application of Limits , so central and dominant is the latter concept . It may be well, to pause again and by way of investigation , determine the meaning of " variable " and "limit " as used by the mathematician. The general notion of the term variable as used by the mathematician , is that a variable is something that varies or changes, like the position of a movable body, the height or size of a growing child , the length of a burning candle, etc. But this notion is entirely wrong ; the error of which can be shown by a few simple examples. One must keep in mind,however, the fact that mathematicians make constant use of figurative speech and employ dynamic terms in describing static facts . It is a common practice among them to use the dynamic term , transformation , ... suggesting change, variation , transmutation, ...to denote what is in fact a static thing, namely, a relation ...... something that is unchanging , eternal . There is a striking incongruity between what is said and what is meant . We shall find that there is just such an incongruity between the mathematical term "variable" and the manner in which the mathematician habitually uses it . This incongruity of speech is permitted bacause it is stimulating and economical, and because it does not, except in certain fundamental questions, lead to error.

Mathematicians speak of the variable as if the mathematical concept of the variable were the concept of something whose essential nature is to suffer change; that is to say , when they use the symbol "x" to denote what they call a variable, they familiarly speak of the variable x as altering its value as increasing or decreasing, as approaching or not approaching a limit, and so on; yet in spite of such a way of speaking , what they really mean by the term "variable " essentially involves no idea of change whatever, as "change " is commonly understood. The mathematical variable x in the equation $3x - 2x - 9 = 0$, appears as a propositional function; when the mathematician says, " I shall let x represent any point on a certain line "L" , thereby indicating that he will use x as a variable, he virtually says, that he will let x represent any one of the verifiers of the propositional function;. x is a point in the line "L". Take a concrete example....... a lady has on her desk three books.... a novel, a classic and a book of poems. She says to her maid, " Bring me a book from my desk, Maria "; Maria asks : "Which one ? " " Any one ",replies the lady. As here used , the phrase " a book " is a variable because it represents "any one " of a certain class of books; in representing "any one " of the class, it does not refer to a particular book, for evidently "any one" is not a description or designation of a particular one of the books; neither does it refer to all of the books cänjunctively ...... the novel, the classic and the poems. Maria is not to bring all three ; it does refer to each of them disjunctively. Maria is to bring eithr the novel or the classic or the book of poems, no matter which one . So it is in the definition : in representing any one of the class of admissible terms for O(x) , x does not refer to a specific one of the terms nor to all of them conjunctively; it refers to each of them disjunctively. It is essential and should now be clear that no idea of variation or change is involved. O (x) being given , it is timeless, and unchanging. X's representation of this " any " is therefore, timeless and unchanging. Thus we see that a given mathematical variable is timeless and unchanging, and when the mathematician speaks of a variable, he has no reference to change or variation in the strict sense.

Since the calculus is based upon the idea of the Limit, it is necessary to have a clear understanding of this term also. The idea of a variable approaching a limit occurs in elementary geometry in establishing a formula for the area of a circle. In finding the area of a circle in plane geometry it is usual to begin by inscribing a regular polygon in the circle. The area of the polygon differs from that of the circle by a certain amount. As the number of sides of the polygon is increased, this difference becomes less and less. Thus we first consider the area of the regular inscribed polygon of "n" sides, and "n " is then assumed to increase indefinitely. The variable then approaches a limit, and this limit is defined as the area of the circle. In this case the "variable "v" (the area of the polygon) increases constantly, and the difference a - v , where "a" is the area of the circle, diminishes and ultimately becomes and remaбns less than any preassigned (value) number, however small.Likewise we may illustrate the conception of limit in the consideratio of the geometrical progression with an unlimited number of terms :

$$I - \frac{1}{2} - \frac{1}{2} - \ldots\ldots\ldots\ldots\ldots ,$$

The sum of the first two terms of this series is $I\frac{1}{2}$, the sum of the first three terms is $I\frac{3}{4}$ and so on . It may be found that the sum of the terms becomes more nearly equal to 2 as the number of the terms become greater. It may be further shown that if any small number "e " is assigned , it is possible to take a number of terms "n" so that the sum of these terms differs from 2 by less than "e " . The number 2 is said to be the limit of the sum of the first n terms of the series. From these illustrations we arrive at the definition of the limit as used by the mathematician in connection with the terms constant and variable. The variable "v" is said to approadh the constant "l " as a limit when the successive values of "v" are such that the numerical value of the difference v - l becomes abd remains less than any preassigned value (positive number), however small. The question of a variable approaching a limit occupied the minds of the greatest thinkers of the 5th century, when we find it mentioned in Zeno's Paradoxes. His arguments involve concepts of continuity, of the infinite and of the infinitesimal, and are as much the subject of debate now as they were in the time of Aristotle. Aristotle gave no satisfactory explanation of Zeno's Arguments, but Antiphon , a sophist and contemporary of Hippocrates , in the course of time introduced the process of exhaustion for the purpose of solving the problem of quadrature. The Method of Exhaustion as first applied, consisted in comparing the area bounded by a given curve with the area of an inscribed or circumscribed polygon whose number of sides is continually increaded , until the area of the one coincides with the other. This process is related to the present calculus through the doctrine of Limits. In the same way the early Greeks compared the surfaces of the sphere, of the cone and of the cylinder with prismatic and pyramidal surfaces. The first notion of the calculus was therefore, among the Greeks and from a geometrical viewpoint.

In defining the Differential Calculus, we say that it is the science which deals with the rate at which variable quantities increase or diminish. When we say that a quantity is a variable, we imply that it varies as some other quantity changes. For example, the velocity of a train is variable. It varies with the time which has elapsed since the train started...... it varies with the distance traversed ... with the steam power employed .... with the state of the rails... and so on. But the differential calculus deals only with those quantities which vary according to some definite law. For example, when a body is let fall from rest, the distance it traverses varies, according to known law, with the time elapsed since the fall began, the sine of an angle varies according to a known law as the angle changes ; and differential calculus is therefore, able to deal with both of these specific cases.

Now we can see the value ofa a system which will deal with variable quantities. Algebra and Geometry and Trigonometry deal with absolute quantities....... but it is often necessary to know and learn something about the variation of quantities, to know when a variable quantity attains its greatest value; when it is increasihg , when it is diminishing ; when it changes fastest, and so on. The great advantage to be found in the use of the differential calculus is that it will solve our problem systematically . An ingenious application of Algebra or Geometry or Trigonometry will often enable us to solve the problems which belong especially to the differential calculus, but we require ingenuity for this purpose, whereas the differential calculus solves the problem with certainty; even if we hanen't a particle of ingenuity, so long as we follow the proper rules. Even where it fails, it teaches us that we are trying to solve an insoluble problem. The first matter the calculus attends to is the choice of a convenient expression for the rate at which the variable quantity changes. This expression is is called a differential <u>Coefficient</u> . Whem there is a quantity whose value depends upon some variable, the Differential Coefficient of the quantity with respect to that variable represents the rate at which the quantity varies as the variable changes in value. Secondly, when we know the rate at which a quantity depending on some variable changes with the change of the variable..... in other words. when we know the differential coefficient of the quantity with respect to that variable.... we can determine the quantity itself (called the Integral) , if only we know what quantity it is which has that differential coefficient. Thus while Algebra, Geomtry and Trigonometry deal with quantities whose value is fixed , the calculus investigates quantities whose value is constantly changing. Considering that all nature in all its aspects varies wontinually , the importance of a mathematical method of dealing with variables is evident; and it is easy to see why science had made so little progress befpre the invention ofTthe calculus , and why progress has been so rapid since.

The Calculus , as a consistent method capable of general application was first formulated by Newton and Leibnitz in the latter half of the I7th century. But historical research shows that this period is rather the climax in evolution of the science, and that the process of its development has been both static and dynamic , and in four different steps. Beginning in the 5th century B.C. among the Greeks we have an intimation of integration through the process of exhaustion used by them in treating incommensurable magnitudes. The " Methoã of Infinitesimals ", which made its first appearance in the beginning of the I7th century , through the work of Kepler I6I6 , of Cavalieri I635, Of Descartes I637, of Fermat I638, and of Wallis I655, marks the second general step in the development, which was seized upon by Newton and Leibnitz some time after and budded forth in a third stage known as the " Method of Fluxions " and the"Calculus summatorius ". The fourth stage is that of Limits, the arithmetical conception of which was created by Wallis, and later extended by Newton and other great mathematicians of the period.

The evolution of the science of the Calculus also owes much to the cultivation of philosophy...... which caused men to look beyond the mere evidence of their senses and cultivate ideas of abstract truth...... for in the early history of geometry we note that the science was confined to the solution of such simple propositions as could be seen and verified by experiment. The measurement of rectangles led to that of parallelograms and triangles, rectangular and oblique parallelopipeds, and from thence to prisms with triangular and polygonal bases. But when these crude people came to figures bounded by curved surfaces....... the primitive metho of triangular or rectangular basis failed them . What was to be done....... some one had to make bold and present a solution ...... but who had the ability and courage to solve the difficulty ? Unfortunately this hero's name has been swallowed up in oblivion . The idea of invention was seized upon by Euclid and Archimedes after a time and it is here we find the first records, and note the rapid development of mathematical knowledge and the opening of a brilliant career for the science of Geometry..... which endured for a period of two thousand years without further development. The method used by Euclid and Archimedes was known as the " Method of Exhaustion" often called the " reductio ad absurdum " because it shows that every supposition leads to an absurdity except the true one. Although the method of exhaustion was intro-duced by Antiphon (430 B.C.) , it was Eudoxus who placed the theory on a firm basis in 370 B.C. We have here a crude ap-proach to the integration of the I7th century, but the near-est approach to actual integration was effected in 225 B.C. by Archimedes through his summation of an infinite series. He states in his proof that the area of a parabolic segment is 4/3 of the triangle with the same base and vertex, or 2/3 of the circumscribed parallelogram.

He arrived at these conclusions by continually inscribing in each segment between the parabola and the in-scribed figure a triangle with the same base and height as the segment. Then letting A represent the area of the orig-inal inscribed triangle, he obtained the summation of the series thus :

$A + \frac{1}{4}A + (\frac{1}{4})^2 A + \ldots\ldots$ or finding the value of

$A\left[I + \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + (\frac{1}{4})^4 + \ldots\ldots\right]$, so that the

the process was the same as our present integration. He rec-ognized the fact that $(\frac{1}{4}) \to 0$ as $n \to \infty$ but did assert it.

A manuscript, was found in I906, by Professor Heiburg, the editor of the works of Archimedes, in which were given his conclusions regarding the quadrature of the parabola. Conclusions drawn by Archimedes in the treatment of solids bounded by curved lines (or surfaces), if expressed in modern formulas, would read thus:

The surface of a sphere ....., $4\pi a^2 \cdot \frac{1}{2}\int_0^{\pi} \sin \phi d\phi = 4\pi a^2$

" " " " spherical segment....,

$$\pi a^2 \int_0^{a} 2 \sin \phi d\phi = 2\pi a^2 \ (I - \cos \chi)$$

The volume of a segment of a spheroid.....,

$$\int_0^{b} x^2 dx = I/3\, b^3$$

The volume of a segment of a hyperboloid of revolution..,

$$\int_0^{b} (ax + x^2) dx = b^2 \left(\tfrac{1}{2} a + I/3 \ b \right)$$

The area of a spiral, ... $\frac{\pi}{A}\int_0^{a} x^2 dx = I/3\, \pi a^2$

The area of a parabolic segment,

$$\frac{1}{A^2}\int_0^{\pi} \Delta^2 d\Delta = I/3\ A$$

It was, therefore, by the method of exhaustion that the ancients demonstrated all the intricate problems in elementary geometry, and brought that science to the condition in which it remained for two thousand years........ Truths were waiting on all sides to be discovered, and continued to wait for centuries until a more powerful instrument of discovery could be invented . Kepler was the first to improve on this method by introducing into geometry yhe idea of infinity. He considered the circle as composed of an infinite number of triangles with their vertices at the centre and with their bases on the circumference ; and the cone as composed of an infinite number of pyramids, the summation of which became the problem of later integration. Kepler's attempt at integration caused Cavalieri in I635 to develop his method of Indivisibles, by which he effected quadrature by summing the infinitesimal elements into which he divided his areas.

The only traces we have as an approach to the calculus in the Middle Ages are those of mensuration. Although Oresme' method of latitudes and longitudes in I360 gave rise to what we now call the distribution of the curve or graph -- a step fundamental to the modern method of findind the area included between a curve and certain straight lines.

The period of stagnation ended, however , in I637, by the discoveries of Descartes, who seizing upon a new idea.. dedeveloped a system whose results were both astonishing and delightful. Breaking away from the idea of determinate values and absolute conditions , he adopted that of dependent conditions and relative values, but gave them wide range , which no longer fixed unchangingly the quantities sought. They were called variables, while those quantities whose values were fixed were called constants. Descartes' method was two- fold he introduced the idea of two variables in one equation with a range of similtameous values, These being adapted to each other, united to form a method of investigation, which in a simple and easy manner solved questions which had taxed the powers of the ancients. Kepler's notion that all magnitudes are made up of infinitely small parts, and Cavalieri's modified " Method of Indivisibles " seemed to have caused a great stir in the mathematical world of that period. It brought the greatest genius of all times, B. Pascal and also Roberval to whom we owe the profound research concerning the cycloid, into the field.

" We may get an idea of Cavalieri's method by considering his comparison of a triangle with a parallelogram having the same base and altitude. Calling the smallest element of the triangle I, the next 2, and the next 3, and so on to "n", the base.

The area is, therefore,

$$I + 2 + 3 + \dots\dots\dots\dots + n,$$

or $\frac{1}{2} n (n + I)$, but each element of the parallelogram is n, and there are n of them, as in the triangle, so the area is N. The ratio of the area of the triangle to the area of the parallelogram is

$$\tfrac{1}{2} n (n + I) : N \quad \text{or} \quad \tfrac{1}{2} (I + I/n).,\ \text{but}$$

$\frac{1}{2} (I + I/n) \longrightarrow \frac{1}{2}$ as $n \longrightarrow \infty$, hence the triangle is half of the parallelogram."

(Smith's History of Mathematics Volume II)

It was by this method that Cavalieri solved simple problems in mensuration involving lengths, areas and volumes. Roberval pursuing a similar hypothesis, considered the area between a curve and a straight line as made up of an infinite number of infinitely narrow rectangles strips, whose sum gave the required area. He also applied this method to problems of ratification and of cubature, taking " m " as a positive in* teger, he found the approximate value of ,

$\int_0^1 x^m dx$, , by finding the value of

$$\frac{0^m + I^m + 2^m + \ \ldots\ldots\ldots + (n + I)^m}{n^{m+1}},$$ stating

that this result approach $I/m - I$ as $n \to \infty$

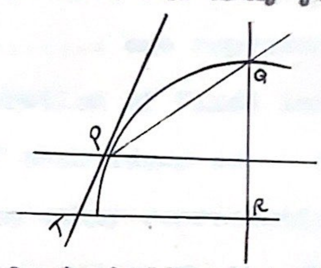
The following year Fermat reached the same conclusions, using Archimedes method as a basis, but extended his proof to include "m " as a fractional and as a negative exponent. He attacked the problem of maxima and minima at the same time ; finding the point on a curve at which a tangent is parallel to the x - axis. Considering these achievements, many authorities hail Fermat as the first inventor of the new calculus.

" In his method De maximus et minimis, he equates the quantity of which one seeks the maximum and minimum to th expression of the area- same quantity in which the unknown is increased by the indeterminate quantity. In this equation ..... he divides by the indeterminate quantity which occurs in them as a factor , then he takes this quantity as zero thus he has an equation which serves to determine the unknown sought........ ....
His method of tangents depend upon the same principle.
The basic principle of the theory of maxima, may be said to go back to Pappus (C300) It appeared indirectly in the works of Oresme I360, thus giving a claim to the Middle Ages.

But it was Fermatwho first stated substantially the law as we recognize it today, thus communicating to Descartes in the year I638 a method which is essentially the same as the one used at present; that of equating

f (y) to zero "

In the year I663 Barrow comes forward with his method of tangents, which is considered as a process of differentiation . In his *Lectiones opticea et geometricae* he lets Q approach P as in our present theory, the result being an infinitely small (infinite parvum) arc. For many years the triangle PQR was known as Barrow's *differential* triangle. This name is not due to him , however.



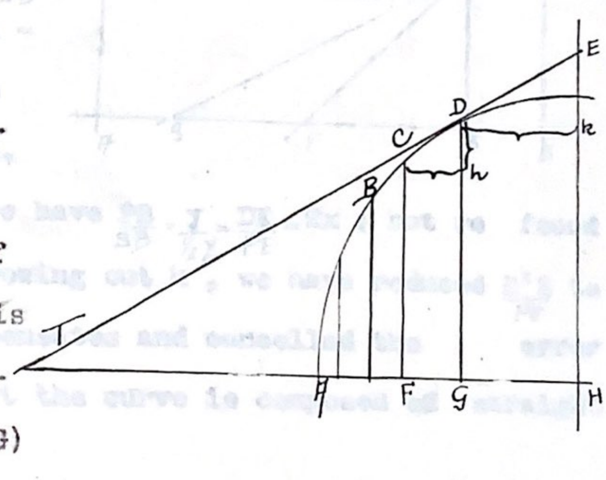
Barrow's method and differential triangle is believed to have influenced the mathematicians of his time. It seems quite probable that as early as I664 , Barrow had informed Newton of his accomplishments. Pascal had published some time before a figure similar to his. The consequence was that the study of triangles of the general nature were being discussed in both England and France. The triangles of Barrow and Pascal were thus probably known to Leibnitz and perhaps influenced him in the devalopment of his own theory. Barrow recognized the fact that integration is the inverse of differentiation, but he did not use this relation in solving his quadrature preblem.

The next movement in the mathematical world occurred when Newton and Leibnitz brought out their systems ,in which the results are essentially the same, although the fundamental idea or philosophy of each is entirely different from the other. Thus we approach what is popularly thought to be the period in which the calculus was invented. Yet it is evident that a crude integral was already in use, and that some approach had been made to the process of differentiation It is also evident that the lines of approach to the calculus in general have been two in number.......... one representing the static phase as seen in the mensuration of fixed lengths areas, or volumes and in making use of such ideas as those of infinitesimals and indivisibles; the other representing th dynamic phase as seen in the motion of a point. To the former belong to such names as Kepler, Cavalieri, Wallis and Archimedes............ These covering the Greek and Mediaval periods of development, and through Pascal and Barrow , we ha have a transition into the I7th century of renown, to which belong the great leaders of mathematics of time of Leibnitz and Newton.

In I684 Leibnitz first published his Differential Calculus in which he gives the required operations.His method was formed on a conception similar to that of Pascal , viz. that all quantities were composed of infinitesimals. But Leibnitz carried his idea much further than Pascal; for he consideredeach of the infinitesimals of his system as also composed of an infinite number of parts , infinitely smaller than itself ; and these again , composed of an infinite number of parts, infinitely smaller yet; and so on, indefinitely. He called them infinitesimals of the first , second , and third, order, and so on , into which each particle or portion of any order infinitely less than the one preceding ; and bore the same relation to it that the infinitesimal of the first bore to thee original function. He indicated these infinitesimal increments, parts or differences , by the term "differentials." Thus we find in his system differentials of the first, second, and third order and so on, which he called " first differential ", " second differential " and so on. The peculiarity of Leibnitz's method may be illustrated thus :

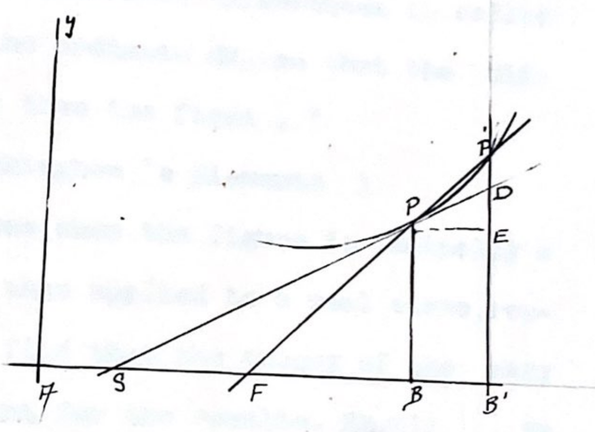
Suppose a curve to be composed of an infinite number of right lines, infinitely short, which for the sake of perspicuity we shall represent by the polygon ABCDE .

" Let CF and DG be consecutive ordinates of the curve CD, an infinitesimal side lying between them. The infinitesimal difference Dh between the ordinates is called the differential of the ordinate C, and this is divided by the corresponding differential Ch or (FG) of the abscissa will give



the tangent of the angle CTF, which the side CD, or the tangent CT, PB by y , and P'Bb by y', we shall have $y = x$ and $y' = (x-h)^2$ whence we derive $\frac{y' - y}{h} = 2x + h$, which represents the tangent of the angle PFB . But the true tangent of the angle made by the tangent line to a parabola like this is equal to 2x , hence the prolongation PP' is not tangent to the curve, and PP' is not , therefore, a part of it . In considering the curve as made up of straight lines, we commit an error, which may be corrected by reducing h to zero in the equation $\frac{y' - y}{h} = 2x + h$, and thus bringing the two points P and P' together, we thereby reduce the cord (or side of the curve) PP' to zero at the same time.

The error arising from taking PP' as a part of the curve is exactly compensated by throwing out h. For if SD is the true tangent line, then instead of P'B we should have $\frac{DE}{PE} = \frac{PB}{SB}$ for the tangent of the angle made by the line with the axis of the abscissas.



And since $SB = \frac{AB}{2} = \frac{X}{2}$, we have $\frac{PB}{SB} = \frac{y}{\frac{1}{2}x} = \frac{DE}{PE} = 2x$; but we found $\frac{PE'}{PE} = 2x + h$, so by throwing out h , we have reduced $\frac{D'E}{PE}$ to $\frac{DE}{PE}$ and this exactly compensates and cancelled the error arising from assuming that the curve is composed of straight lines.

" Two rays of light sometimes make darkness, but it is reserved to the mathematician to exhibit the phenomenon of two rays of darkness producing light ."

which is the prolongation of the side CD, makes with the axis of the abscissas . To ascertain whether the curve is convex or concave toward the axis of the abscissa, the next infinitesimal DE of the curve is taken , and the infinitesimal Ek between its two ordinates is compared with the infinitesimal difference between the ordinates of the preceding side CD, and the difference between these two infinitesimal differences will show which way the curve is bending. If this difference is negative the second infinitesimal Ek is less than the first, and the curve is concave toward the axis of the abscissas ; if it is positive , the last infinitesimal differ- is greater, and the curve is convex toward the axis. The difference between these two infinitesimal differences is called the second differential of the ordinate CF, so that the differential is infinitely less than the first ."

(Buckingham 's Elements)

Now this method of differences when the figure is actually a apolygon is reasonable, but when applied to a real curve, represented by an equation, we find that the theory of the many sided polygon will not account for the results. Should we take for example the parabola APP' of which the equation is $y = x^2$, and suppose it to be composed of an infinite number of infinitely small straight lines. Let PP' stand as one of these sides ; then representing AB by x, BB' by h,